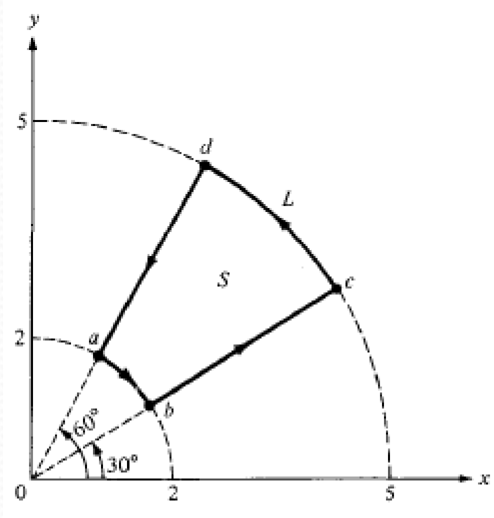
LECTURE NO 12

If $\mathbf{A} = \rho \cos\phi\, \mathbf{a}_\rho + \sin\phi\, \mathbf{a}_\phi$, evaluate $\oint \mathbf{A} \cdot d\mathbf{l}$ around the path shown in Figure 3.22. Confirm this using Stokes's theorem.



Solution:

Let

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \left[\int_a^b + \int_b^c + \int_c^d + \int_d^a \right] \mathbf{A} \cdot d\mathbf{l}$$

where path L has been divided into segments ab, bc, cd, and da as in Figure 3.22.

Along ab, $\rho = 2$ and $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$. Hence,

$$\int_a^b \mathbf{A} \cdot d\mathbf{l} = \int_{\phi=60^\circ}^{30^\circ} \rho \sin\phi\, d\phi = 2(-\cos\phi) \Big|_{60^\circ}^{30^\circ} = -(\sqrt{3} - 1)$$

Along bc, $\phi = 30°$ and $d\mathbf{l} = d\rho\, \mathbf{a}_\rho$. Hence,

$$\int_b^c \mathbf{A} \cdot d\mathbf{l} = \int_{\rho=2}^{5} \rho \cos\phi \, d\rho = \cos 30° \, \frac{\rho^2}{2} \Big|_2^5 = \frac{21\sqrt{3}}{4}$$

Along cd, $\rho = 5$ and $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$. Hence,

$$\int_c^d \mathbf{A} \cdot d\mathbf{l} = \int_{\phi=30°}^{60°} \rho \sin\phi \, d\phi = 5(-\cos\phi) \Big|_{30°}^{60°} = \frac{5}{2}(\sqrt{3} - 1)$$

Along da, $\phi = 60°$ and $d\mathbf{l} = d\rho\, \mathbf{a}_\rho$. Hence,

$$\int_d^a \mathbf{A} \cdot d\mathbf{l} = \int_{\rho=5}^{2} \rho \cos\phi \, d\rho = \cos 60° \, \frac{\rho^2}{2} \Big|_5^2 = -\frac{21}{4}$$

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = -\sqrt{3} + 1 + \frac{21\sqrt{3}}{4} + \frac{5\sqrt{3}}{2} - \frac{5}{2} - \frac{21}{4}$$

$$= \frac{27}{4}(\sqrt{3} - 1) = 4.941$$

Using Stokes's theorem (because L is a closed path)

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

But $d\mathbf{S} = \rho \, d\phi \, d\rho \, \mathbf{a}_z$ and

$$\nabla \times \mathbf{A} = \mathbf{a}_\rho \left[\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] + \mathbf{a}_\phi \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] + \mathbf{a}_z \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right]$$

$$= (0 - 0)\mathbf{a}_\rho + (0 - 0)\mathbf{a}_\phi + \frac{1}{\rho}(1 + \rho) \sin \phi \, \mathbf{a}_z$$

Hence:

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_{\phi = 30^\circ}^{60^\circ} \int_{\rho = 2}^{5} \frac{1}{\rho} (1 + \rho) \sin \phi \, \rho \, d\rho \, d\phi$$

$$= \int_{30^\circ}^{60^\circ} \sin \phi \, d\phi \int_2^5 (1 + \rho) d\rho$$

$$= -\cos \phi \, \Big|_{30^\circ}^{60^\circ} \left(\rho + \frac{\rho^2}{2} \right) \Big|_2^5$$

$$= \frac{27}{4} (\sqrt{3} - 1) = 4.941$$